net returns $\quad R_t = \dfrac{P_t}{P_{t-1}} - 1 \qquad P_t$ price of the asset at time t

gross returns $\quad 1 + R_t = \dfrac{P_t}{P_{t-1}}$

gross returns over the most recent k periods

$$* \quad \boxed{1 + R_t(k)} = (1+R_t)(1+R_{t-1}) \cdot \ldots \cdot (1 + R_{t-k+1})$$

ending period $\qquad$ holding period

No statistical assumptions are made so far.
No dividends. (nonstatistical assumption)
No randomness involved.

Log returns

$$r_t = \log(\underbrace{1+R_t}_{\text{gross return over 1 period}})$$

$$\overset{\text{def of gross returns}}{=} \log\left(\frac{P_t}{P_{t-1}}\right) = \underbrace{\log P_t}_{(\text{lowercase})\ p_t} \underbrace{- \log P_{t-1}}_{p_{t-1}}$$

← not used very often in book

$\log(1+x) \simeq x$ for x small enough

$$\log(1+x) \leq x$$

A consequence of this is that $\log(1+R_t) \simeq R_t$

$$r_t(k) = \log(1+R_t(k))$$

$$\overset{\text{mult. becomes addition}}{=} \log(1+R_t) + \log(1+R_{t-1}) + \ldots + \log(1+R_{t-k+1})$$

$$= r_t + r_{t-1} + \ldots + r_{t-k+1}$$

Exercises 4, 5, 6

The r_t's are independent random variables. (RW hypothesis)

r_t's independent r.v.'s

$N(\mu, \sigma^2)$ common distribution

normal with mean μ and variance σ^2

(indexed by parameters μ and σ^2)

$$r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1}$$

sum of independent r.v.'s

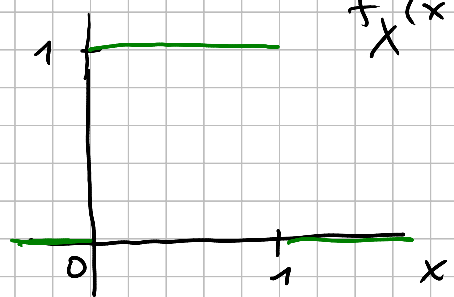
identical distribution $N(\underline{\mu}, \underline{\sigma^2})$

$r_t(k)$ is known to be $\underline{N}(\_\_, \_\_)$

WARNING X, Y have same dist. but it does not necessarily guarantee that $X+Y$ will share the same dist.

$X \sim U(0,1)$ pdf (prob. density function)

$$f_X(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$



$X_1, \dots, X_n$ same dist $\Rightarrow X_1 + \dots + X_n$

(reproductive property)

$$* \ r_t(k) = \underline{r_t + r_{t-1} + \cdots + r_{t-k+1}} \to N(\underset{?}{k\mu}, \underset{?}{k\sigma^2})$$

independent of each other

common dist. $N(\mu, \sigma^2)$

$\mu = \mathbb{E}(r_j)$

$j = t, \dots, t-k+1$

$\sigma^2 = Var(r_j)$

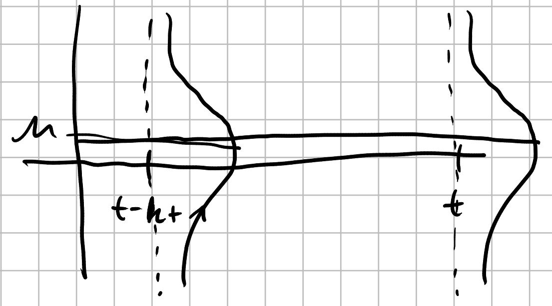
$$\mathbb{E}(r_t(k)) = \mathbb{E}(r_t + r_{t-1} + \cdots + r_{t-k+1})$$

expected value of a sum of r.v.'s is sum of expected values of each r.v.

$$\overset{\leq}{=} \mathbb{E}(r_t) + \mathbb{E}(r_{t-1}) + \cdots + \mathbb{E}(r_{t-k+1})$$ returns

$$= \mu + \mu + \cdots + \mu$$

$$= k\mu$$



$$Var(r_t(k)) = Var(r_t + r_{t-1} + \cdots + r_{t-k+1})$$

variance of a sum is sum of variances if the r.v.'s are independent of each other!

$$\overset{\downarrow}{=} Var(r_t) + Var(r_{t-1}) + \dots + Var(r_{t-k+1})$$

$$\underset{\uparrow}{=} \sigma^2 + \sigma^2 + \dots + \sigma^2 = k\sigma^2$$

identical

<u>NOTE</u> If X & Y are r.v.'s (not necessarily independent),

$$Var(X+Y) = Var(X) + Var(Y) + 2\underline{Cov(X,Y)}$$

Conclusion: $\boxed{\begin{array}{l} r_t(k) \sim \underline{N(k\mu, k\sigma^2)} \\ \text{under assumptions that } r\text{'s are independent and} \\ \text{have a common } N(\mu, \sigma^2) \text{ distribution!} \end{array}}$

X and Y are independent of each other

if and only if $\underline{f_{X,Y}(x,y)} = \boxed{f_X(x)}\,\boxed{f_Y(y)}$ (continuous)

$\underset{\uparrow}{\underline{\mathbb{P}(X=x, Y=y)}} = \boxed{\mathbb{P}(X=x)}\,\boxed{\mathbb{P}(Y=y)}$ (discrete)

joint distribution — marginal dist. of X — marginal dist of Y

another way to see this

$$\mathbb{P}(Y=y \mid X=x) = \underline{\mathbb{P}(Y=y)}$$

$$\mathbb{P}(X=x \mid Y=y) = \mathbb{P}(X=x)$$

$$\mathbb{P}(Y=y \mid X=x) > \mathbb{P}(Y=y) \rightarrow \text{dependence.}$$

$Z_1, Z_2, \dots$ $\underbrace{IID(\mu, \sigma^2)}_{\text{Noise (Shocks)}}$

$S_1 = S_0 + Z_1 \quad t=1$

$S_2 = S_1 + Z_2 \quad t=2$

$= S_0 + Z_1 + Z_2$

$\vdots$

$S_t = S_0 + \underline{Z_1} + \dots + \underline{Z_t}$

Conditional expectation CEF

$\mathbb{E}(S_t \mid S_0)$

$= \mathbb{E}(S_0 + Z_1 + \dots + Z_t \mid S_0)$

$= S_0 + \mathbb{E}(Z_1 + \dots + Z_t \mid S_0)$

$\overset{*}{=} S_0 + \mathbb{E}(Z_1 + \dots + Z_t)$

shocks are indept of the starting point

$$= S_0 + \underbrace{\mu + \cdots + \mu}_{t}$$

$$\Rightarrow \mathbb{E}(S_t | S_0) = S_0 + \mu t \qquad = S_0 + \mu t$$

$$\Rightarrow Var(S_t | S_0) = \sigma^2 t \qquad \left(\begin{array}{l} Var(S_1|S_0) = \sigma^2 \\ Var(S_2|S_0) = 2\sigma^2 \end{array} \right.$$

Empirical rule for normal dist

$X \sim N(\mu, \sigma^2)$

prob X is within 1 s.d. of μ.

$$\mathbb{P}(\mu - \sigma \le X \le \mu + \sigma) \approx 68\%$$

$$\mathbb{P}(\mu - 2\sigma \le X \le \mu + 2\sigma) \simeq 95\%$$

$$\mathbb{P}(\mu - 3\sigma \le X \le \mu + 3\sigma) \simeq 99\%$$

$$S_t | S_0 \sim N(S_0 + \mu t, \sigma^2 t) \qquad \mathbb{P}(S_0 + \mu t - \sigma\sqrt{t} \le S_t \le S_0 + \mu t + \sigma\sqrt{t}) \simeq 68\%$$

$$P_t = P_0 \exp(r_t + r_{t-1} + \cdots + r_1)$$

$$\underbrace{\log P_t}_{S_t} = \underbrace{\log P_0}_{S_0} + \underbrace{r_t + r_{t-1} + \cdots + r_1}_{Z's}$$

Exercise 1

wealth after 1 day? $\rightarrow 1000(1 + \underset{\text{gross return}}{R_t})$

(a) $\mathbb{P}(1000(\underbrace{1+R_t}_{\log r}) < 990) = ?$ What is dist of R_t?

$\rightarrow \mathbb{P}(r_t < \log(0.99))$

$$1000(1+R_t) < 990$$

$$\Rightarrow 1 + R_t < 0.99$$

$$\Rightarrow \underbrace{\log(1+R_t)}_{} < \log(0.99)$$

$$r_t < \log(0.99)$$