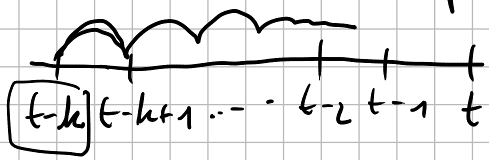
P_t price of an asset at time t

R_t net returns $\quad R_t = \dfrac{P_t}{P_{t-1}} - 1 = \dfrac{P_t - P_{t-1}}{P_{t-1}}$ $\quad P$ uppercase

$1+R_t$ gross return $\quad 1+R_t = \dfrac{P_t}{P_{t-1}}$

$1+R_t(k)$ gross return over the most recent k periods

ending period (t), holding period (k)



$$\underline{1+R_t(k)} = (\underline{1+R_t})\cdot(\underline{1+R_{t-1}})\cdot \dots \cdot(\underline{1+R_{t-k+1}})$$

r_t log returns $\quad \underline{r_t} = \underbrace{\log(1+R_t)}$ log of gross returns

$$\log(1+x) \simeq x \qquad x \text{ is small enough}$$

$$r_t = \log(1+R_t) \approx R_t \qquad \text{if } R_t \text{ is small enough}$$

$$r_t \leq R_t$$

$$r_t = \log(1+R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = \underbrace{\log P_t}_{p_t} - \underbrace{\log P_{t-1}}_{p_{t-1}}$$

$\leftarrow$ not used often in the book

$$\underline{r_t(k)} = r_t + r_{t-1} + \dots + r_{t-k+1}.$$

log returns over k periods = sum of log returns each period.

- No statistical assumptions made.
- No dividends. (nonstatistical assumption)

Random walk model

r_t's are independent random variables
log returns

The r.v.'s X & Y are independent if and only if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \text{(continuous case)}$$

$$\underbrace{\mathbb{P}(X=x \,\&\, Y=y)}_{\text{joint distribution}} = \mathbb{P}(X=x)\,\mathbb{P}(Y=y) \quad \text{(discrete case)}$$

involves r.v. X only (marginal dist of X)

involves r.v. Y only (marginal dist of Y)

another way of seeing this discrete case:

$$\mathbb{P}(Y=y \mid X=x) = \mathbb{P}(Y=y)$$

$$\underbrace{\mathbb{P}(X=x \mid Y=y)}_{\text{conditional distribution}} = \mathbb{P}(X=x)$$

$$\mathbb{P}(Y=y \mid X=x) > \mathbb{P}(Y=y)$$ some dependence between X & Y.

---

* $r_t(k) = r_t + r_{t-1} + \cdots + r_{t-k+1} \sim$ dist? $N(?,?)$

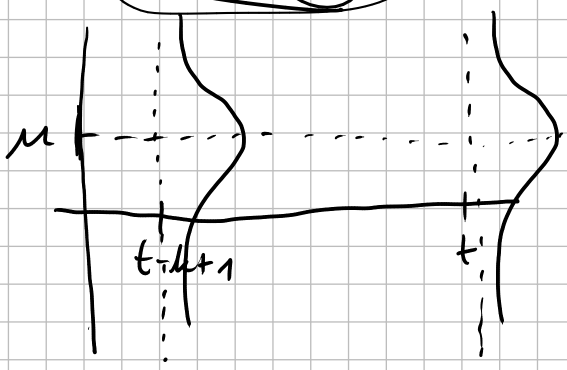
r_t's independent of each other

share a common dist $N(\mu, \sigma^2)$

$\mathbb{E}(r_j) = \mu$

$j = t-k+1, \ldots, t$

$\mathrm{Var}(r_j) = \sigma^2$



WARNING If X, Y have the same dist, it does not automatically mean that X+Y will have the same shape of the dist of X & Y.

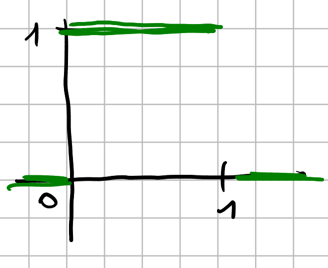
$X \sim U(0,1)$

$Y \sim U(0,1)$

$X+Y \sim ?$

pdf (prob. density function)

$$f_X(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$r_t(k) = r_t + \dots + r_{t-k+1} \sim N(\underbrace{\overset{k\mu}{?}}_{\mathbb{E}(r_t(k))}, \underbrace{?}_{Var(r_t(k))})$$

$$\mathbb{E}(r_t(k)) = \mathbb{E}(r_t + r_{t-1} + \dots + r_{t-k+1})$$

$r_j \sim N(\mu, \sigma^2)$

$j = t-k+1, \dots, t$

expected value of a sum is the sum of the expected values.

$$= \mathbb{E}(r_t) + \mathbb{E}(r_{t-1}) + \dots + \mathbb{E}(r_{t-k+1})$$

$$= \underbrace{\mu + \mu + \dots + \mu}_{k \text{ of them.}}$$

$$= k\mu$$

$$Var(r_t(k)) = Var(r_t + r_{t-1} + \dots + r_{t-k+1})$$

variance of a sum is the sum of variances $\boxed{\text{if}}$ the r.v.'s are independent of each other

$$= Var(r_t) + Var(r_{t-1}) + \dots + Var(r_{t-k+1})$$

share a common dist

$$= \underbrace{\sigma^2 + \sigma^2 + \dots + \sigma^2}_{k \text{ of them}}$$

$$= k\sigma^2$$

CONCLUSION :

$$r_t(k) = r_t + r_{t-1} + \dots + r_{t-k+1} \sim N(k\mu, k\sigma^2)$$

under the random walk hypothesis + common $N(\mu, \sigma^2)$ return distribution

NOTE X, Y r.v.'s (not necessarily independent)

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X, Y)$$

$Z_1, Z_2, \dots \sim IID(\mu, \sigma^2)$ (noise / shock)

$t=0 \quad S_0$

$t=1 \quad S_1 = S_0 + Z_1$

$t=2 \quad S_2 = S_1 + Z_2 = S_0 + Z_1 + Z_2$

$\vdots$

$$S_t = S_0 + Z_1 + \dots + Z_t$$

conditional expectation / CEF

$\mathbb{E}(S_t | S_0) = S_0 + \mu t$

cond. variance

$Var(S_t | S_0) = \sigma^2 t$

$\downarrow$

$Var(S_1 | S_0) = \sigma^2$

$Var(S_2 | S_0) = 2\sigma^2$

$S_t = S_0 + \underbrace{Z_1 + Z_2 + \dots + Z_t}$

$\mathbb{E}(S_t | S_0)$

$\uparrow$ treated as constant

$= \mathbb{E}(S_0 + Z_1 + \dots + Z_t | S_0)$

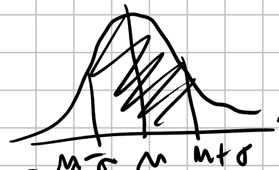
$= \mathbb{E}(S_0 | S_0) + \mathbb{E}(Z_1 + \dots + Z_t | S_0)$

steps are indept of the starting point $\rightarrow$ $= S_0 + \mathbb{E}(Z_1 + \dots + Z_t)$

$= S_0 + \underbrace{\mu + \mu + \dots + \mu}_{t \text{ of them}}$

$= S_0 + \mu t$

Empirical rule for normal dist.



$X \sim N(\mu, \sigma^2) \implies \mathbb{P}(\mu - \sigma \leq X \leq \mu + \sigma) \approx 68\%$

$\mathbb{P}(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx 95\%$

$\mathbb{P}(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx 99\%$

normal random walk

$S_t | S_0 \sim N(S_0 + \mu t, \sigma^2 t) \implies \mathbb{P}(S_0 + \mu t - \sigma\sqrt{t} \leq S_t \leq S_0 + \mu t + \sigma\sqrt{t})$

$\approx 68\%$

$\vdots$

(2.7)

$P_t = P_0 \exp(r_t + r_{t-1} + \dots + r_1)$

$\underbrace{\log P_t}_{S_t} = \underbrace{\log P_0}_{S_0} + \underbrace{r_t + r_{t-1} + \dots + r_1}_{Z_t + Z_{t-1} + \dots + Z_1}$ (2.5)

Ex 4,5,6 only use Section 2.1

Exercise 1 \$1000

wealth at end of 1 trading day $\quad 1000(\underbrace{1 + R_t}_{\text{gross return}})$

$\mathbb{P}(1000(1 + R_t) < 990) \longrightarrow$

$$1000\,(1+R_t) < 990$$

$$\Rightarrow \quad 1+R_t < 0.99$$

$$\Rightarrow \quad \underbrace{\log(1+R_t)}_{r_t} < \log(0.99)$$

$$\mathbb{P}\left(r_t < \log(0.99)\right)$$